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The Clock Paradox in the Relativity Theory

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Abstract

A system S' (rocket) starts from rest in an inertial system S, and after a series of accelerated, uniform and decelerated motions, comes back to rest at its initial position in S. An exact calculation is carried out, from the standpoint of S, of the time intervals for the arrivals at S of light signals sent back by S'. From the standpoint of S', S has made a round trip after undergoing a series of free falls in gravitational fields and coasting motions. An exact calculation is carried out for the 'proper time' intervals in S from the standpoint of S'. It is shown that there is exact agreement between S and S' in their reckonings of the total time intervals for the two frames, namely, both S and S' agree quantitatively, to them, the time interval is longer for S than for S'.

The accelerated motion of S' relative to S explicitly used in the treatment of the problem in the present work is that under time-independent field and subject to the condition of *local* Lorentz contraction and dilation; the resulting motion turns out to be that obtained earlier by Møller on entirely different considerations. The result of the present treatment is, however, more general than this particular motion seems to imply, since by an arbitrary coordinate transformation, it can be made to include an infinite number of accelerated frames including time-dependent fields, all within the framework of flat space-time. General remarks are given for the clock problem in the general theory of relativity in the sense of Einstein's curved space.

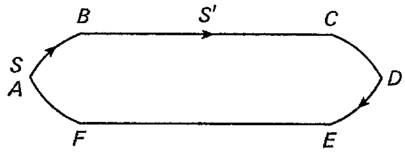
1. *The Clock Paradox*

The question concerned is the following: Imagine a pair of clocks, one of which remains at rest in an inertial frame, and the other sets out on a trip (on a rocket, say), and after a time returns to rest in the inertial frame. Will the travelling clock be slower than the one at home? Will they both agree exactly by how much one is slower than the other?

This problem is sixty years old. In a paper in 1911, Einstein (1911) gave a simple theory in which (1) he employed the Doppler effect formula of the special theory of relativity and obtained the effect of uniform acceleration of a reference frame on the Doppler shift, and (2) he introduced the equivalence principle for the acceleration of a frame and a gravitational field. Einstein concluded that a clock that has travelled, say in a circular path, will 'lose time', because the rate of the clock is slower in the accelerated motion.

That a returned clock should have lost time compared with the one at home is so strange a conclusion that Einstein specifically wrote an article (Einstein, 1918) in 1918, in the form of a dialogue between a critic and himself, to show (1) how the trip will be viewed from the standpoints of both frames, (2) how the reciprocal symmetry (in the sense of the special theory of relativity) will be destroyed in this case by the accelerated motion of the rocket, and that both frames will agree that the returned one will be slow, and (3) that this is due to the loss of time, or the 'slowing down of the clock', of the rocket *during* the accelerated portion of the rocket's trip when it *turns back*.

For definiteness, let us pose the following situation. From the standpoint of the inertial system S, the rocket (or the travelling twin) S' goes through the following sequence of events:



(1.1)

A–B: S' starts off, with acceleration a (in the positive x direction), reaching the velocity v at B.

B–C: S' shuts off its engine and moves with a uniform velocity v relative to S.

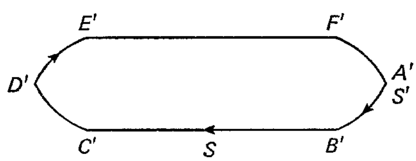
C–D: S' starts its engine and decelerates, reducing its velocity relative to S to zero.

D–E: S' keeps its engine and starts accelerating toward S, reaching the velocity $-v$ at E.

E–F: S' shuts off its engine and moves with constant velocity ($-v$) toward S.

F–A: S' starts its braking engine, and moves with acceleration $-a$, coming to rest at A.

From the standpoint of the rocket S' who regards itself as at rest, S will go through the following events:



(1.2)

A'–B': S starts 'falling' in a (universal) gravitation field $-g$ (in the negative x direction), attaining a velocity $-v$ (relative to S') at B'.

B'–C': The gravitational field is removed, and S keeps on moving with the (constant) velocity $-v$.

C'–D': A (universal) gravitational field g in the positive x-direction is turned on. S comes to stop (relative to S') at D'.

D'–E': The same field g continues to act, and S 'falls' from D' to E', attaining the velocity v (relative to S').

E'–F': The field g is removed and S moves with the constant velocity v.

F'–A': A gravitational field $-g$ is turned on, and S is brought to rest at A'.

During A'–B', C'–D', D'–E' and F'–A', S' itself is in the same universal gravitational field as S, but S' is held fixed by some external agency.

During the uniform relative motion parts B'–C', E'–F', S will say that the clock of S' is slow according to the time dilatation relation. If $\Delta\tau_{S'}$ is the proper time interval recorded by S' for each of these parts, and ΔT_S is the time interval recorded by synchronised clocks attached to the frame S, then

$$2\Delta T_S = \frac{2\Delta\tau_{S'}}{\surd(1-\beta^2)} \tag{1.3}$$

But with equal right, S' will say that the clock of S is slow compared with that of S', and if $\Delta\tau_S$ is the (proper) time interval recorded by S (for each of the uniform relative motion parts) and $\Delta T_{S'}$ the time interval recorded by synchronised clocks attached to the frame S', then

$$2\Delta T_{S'} = \frac{2\Delta\tau_S}{\surd(1-\beta^2)} \tag{1.4}$$

Einstein pointed out, however, that during the 'turning around' parts C'–D', D'–E' in (1.2), S is at a higher gravitational potential than S', and the clock of S is faster than that of S'. The clock in S will during C'–D'' and D'–E' 'gain' time and more than compensate the 'loss' as given by (1.4). If the time intervals during C'–D', D'–E' are very short compared with those for the uniform motion parts, the 'gain' during C'–D', D'–E' by the clock of S will be such as to bring the total time ΔT recorded by S (for the trip B'–C', C'–D', D'–E', E'–F') to be longer than that $\Delta T'$ recorded by S', in accordance with (1.3), i.e.,

$$\Delta T = \frac{\Delta T'}{\surd(1-\beta^2)} \tag{1.5}$$

The above statement of Einstein has been expressed in explicit form by Tolman (1934). Let us view the trip from the standpoint of S' as in (1.2). Let τ_{AB}, τ_{BC}, τ_{CDE}, τ_{EF} $(=\tau_{BC})$, τ_{FA} $(=\tau_{AB})$ be the proper time intervals as recorded by S, and let t_{BC}, t_{CDE}, t_{EF} $(=t_{BC})$, t_{FA} $(=t_{AB})$ be the time intervals as recorded by synchronised clocks at various points in (1.2) attached to S'.† Then, by (1.4),

$$\tau_{BC}\,(=\tau_{EF}) = \surd(1-\beta^2)\,t_{BC}\,(=\surd(1-\beta^2)\,t_{EF}) \tag{1.6}$$

† We shall, without causing confusion, drop the prime for B', C', etc. in the subscripts for the τ's and t's.

Let the average distance between S and S' during the turning around portion C'–D'–E' be approximately taken to be $x = vt_{BC}$. Since $v = gt_{CD}$, the Doppler effect separation gives

$$\tau_{CDE} \cong \left(1 + \frac{gx}{c^2}\right) t_{CDE} \tag{1.7}$$

$$= t_{CDE} + 2\beta^2 \, t_{BC} \tag{1.7a}$$

The total time for the whole trip is, for S, from (1.6) and (1.7a),

$$\begin{aligned} \tau &= \tau_{AB} + 2\tau_{BC} + \tau_{CDE} + \tau_{FA} \\ &= 2[\sqrt{(1-\beta^2)} + \beta^2]\, t_{BC} + 2\tau_{AB} + t_{CDE} \\ &= 2(1 + \tfrac{1}{2}\beta^2 + \cdots)\, t_{BC} + 2\tau_{AB} + t_{CDE} \end{aligned} \tag{1.8}$$

If we make the time intervals τ_{AB}, T_{CDE} very short compared with τ_{BC} and t_{BC}, then (1.8) becomes

$$\tau \cong \frac{1}{\sqrt{(1-\beta^2)}} \times \text{time for the trip recorded in } S' \tag{1.9}$$

This is in approximate agreement with (1.3). It is important to note that the $\simeq$ sign in (1.9) arises not so much because of the neglect of τ_{AB} and t_{CDE} in (1.8) as because of the approximations made in obtaining (1.7).

In 1956, Dingle (1956) in a series of articles renewed the question of whether the returned twin from a rocket trip is younger than his brother who has stayed home. He believed that there should be no difference in their aging, that all earlier conclusions, including Einstein's, are erroneous. His questioning of these earlier works by many physicists has led to a great flux of discussions. Most authors (Arzelies, 1966) maintain the conclusion of Einstein. In most cases, the arguments amount to the simple statement that since the rocket S' has undergone accelerated and decelerated motions, it is not on equal footing with S which is an inertial system, and hence the reciprocal symmetry in the sense of the special theory of relativity has been removed. This part of the argument is of course correct. But then, because of attempts to simplify the problem for the non-specialist, the following argument is usually put forward: One can make the time intervals for the accelerated and decelerated parts very short compared with the time intervals for the uniform relative motion parts [see (1.1) or (1.2)] and in the limit negligible. Then, since only S is a 'preferred' (in the sense that it is an inertial) frame, one must only employ the relation (1.3). This part of the argument is unfortunately misleading. We have seen in the preceding section from the approximate treatment by Tolman that it is precisely the acceleration (or, an equivalent gravitation field) during the 'turning around' of the rocket that 'slows down its clock' (relative to the inertial frame S), and that one obtains the result (1.9) in an *approximation* only, which is *not* exactly the relation (1.3). The point that seems to have been forgotten in many 'elementary' discussions of the clock paradox is that while the 'compensations' in (1.8) and (1.9) must come from the accelerated motions, the correct result [(3.20) and (3.22) in the following] should really be independent of the strength of the accelerating and decelerating field which determine the length of time for these accelerated and decelerated parts. The undue prominence given the uniform relative motion parts (coasting of the rocket) and the consequent appearance of the Lorentz relation (1.3) are unfortunate, for they tend to divert the attention from the accelerated motion, which is essential in the clock paradox problem, to the expression (1.3) for uniform relative motion. In actual fact, the uniform relative motion parts [B–C, E–F, B'–C', E'–F' in (1.1) and (1.2)] are non-essential, and one would have essentially the same 'paradox' if one does away with these (coasting) parts entirely. In the following section, an analysis of the clock problem with, and also without, the uniform relative motion parts in (1.1) and (1.2) will be carried out to illustrate this point.

In the literature, attempts have been made to convince one of the immediate applicability of the expression (1.3) for the time intervals for the whole trip as recorded by S and S' (assuming negligible times for the accelerated parts of the motion) by the following argument. Let there be a set of triplets instead of a pair of twins. Let C stay home (in an inertial frame); let A be moving away in a rocket with velocity v (relative to C). At a certain point in space, A meets B who is travelling toward C with velocity $-v$ (relative to C). A and B do not stop; B just sets his clock according to that of A. B finally passes by C. It is then claimed that the time recorded by C (for the interval between the passing by of A and that of B) is longer than that recorded by B, in accordance with (1.3). This argument is a special case of a general theorem in the special theory of relativity, namely, that in a Minkowski diagram, the time interval measured on a straight line AB (which is the time axis) is longer than the sum of time intervals measured along a series of straight lines AC, CD, . . . DB (each being a time axis in another Lorentz frame) which together with AB form a polygon. The implication of this argument is that we make use of the awareness of the accelerations to remove the reciprocal symmetry of the Lorentz frames, but have ignored the effects of the accelerations on the time measures of the systems. On this argument, one might as well contend with two Lorentz frames since the use of a third frame does not add to the resolution of the problem.

2. *Arbitrary Motion Relative to an Inertial Frame*

We shall study an accelerated motion that can be treated exactly in the clock problem.

Let (X,T) be the space and time coordinates in an inertial frame S and let x,t be those in a frame S' which may be accelerated under the time-independent field. Let $v(x,T)$ be the velocity of a fixed point x in S' relative to S at time T and let X be the space coordinate of the point p so that $X = X(x,T)$ and

$$v(x,T) = v(x(X,T),T) \tag{2.1}$$

The velocity of the point p in S is

$$v = \left(\frac{\partial X}{\partial T}\right)_x \tag{2.2}$$

If we assume that in S' the unit of length is the same as that in S, then the condition of local Lorentz contraction is expressed by

$$\left(\frac{\partial X}{\partial x}\right)_T = \sqrt{\left(1 - \frac{v^2}{c^2}(x, T)\right)} \tag{2.3}$$

From this, one obtains the equation

$$\left(\frac{\partial v(x, T)}{\partial x}\right)_T = \left[\frac{\partial}{\partial T}\sqrt{\left(1 - \frac{v^2}{c^2}(x, T)\right)}\right]_x \tag{2.4}$$

This is equivalent to

$$\left(\frac{\partial v(X, T)}{\partial X}\right)_T = -\frac{1}{c^2}v(X, T)\frac{\partial v}{\partial T}(X, T) \tag{2.5}$$

in which v is regarded as a function of X, T through the transformation (2.1).

To obtain the relation between the time t and the coordinates X and T, we shall introduce the conditions of local Lorentz time dilatation and relativity of motion of S and S', namely,

$$\sqrt{(g_{44})}\, d\tau = \sqrt{\left(1 - \frac{v^2}{c^2}(x, T)\right)}\, dT \tag{2.6}$$

and

$$v(X, T) = \left(\frac{\partial X}{\partial T}\right)_x = -\frac{1}{\sqrt{(g_{44})}}\left(\frac{\partial x}{\partial t}\right)_X = v(x, t) \tag{2.7}$$

where τ is the proper time in S' in the sense that if the metric in S' is $(dy = dz = 0)$

$$ds^2 = -(dx^2 + dy^2 + dz^2) + g_{44}\, dt^2 \tag{2.8}$$

then

$$ds^2 = g_{44}\, d\tau^2 \tag{2.9}$$

so that

$$d\tau = \sqrt{(1 - v^2)}\, dt \tag{2.10}$$

where

$$v^2 = \frac{1}{g_{44}}\left(\frac{\partial x}{\partial t}\right)_X^2 \tag{2.7$'$}$$

Note that the τ called the proper time above and defined by (2.9) is not the normal proper time τ_0 defined by $d\tau_0 = ds$ which will be related to τ here by $d\tau_0 = \sqrt{(g_{44})}\, d\tau$. In the present work, we make use of τ in the calculation of τ_0.

The conditions (2.3) and (2.6) are valid at a point (X, T) or (x, t). Our object is to find a class of accelerated frames S' (with respect to S) with the transformation

$$x = x(X, T), \qquad t = t(X, T) \tag{2.11}$$

and satisfying (2.3) and (2.6). The hypothesis that a transformation (2.11) exists between the (X, T) in an inertial frame and the (x, t) coordinates implies that the space is Euclidean. In this case, we can integrate the differential equations (2.5) and (2.6) since no curvature of space is involved.

Equation (2.5) can be solved by the method of separation of variables, namely, by setting

$$v = \xi(X)\,\zeta(T)$$

which leads to

$$v = \frac{\lambda c^2 T + C_1}{\lambda X + C_2}, \qquad \lambda, C_1, C_2 \text{ being constants.}$$

If the initial condition is

$$v \to aT \qquad \text{at } X = 0 \qquad \text{as } T \to 0$$

then

$$v = \frac{aT}{1 + (aX/c^2)} \tag{2.12}$$

Putting this into (2.2), one obtains the equation of motion of the point p ($x = 0$ in S'),

$$\frac{dX}{dT} = \frac{aT}{1 + (aX/c^2)} \tag{2.13}$$

To obtain the relationship between X and x, we may first integrate (2.13) to obtain

$$\left(1 + \frac{aX}{c^2}\right)^2 - \frac{a^2 T^2}{c^2} = f(x) \tag{2.14}$$

where $f(x)$ is an arbitrary function of x. On the other hand we can also integrate the Lorentz contraction equation (2.3) in which v is given by (2.12). The integration yields

$$\left(1 + \frac{aX}{c^2}\right)^2 - \frac{a^2 T^2}{c^2} = \left(\frac{ax}{c^2} + b(T)\right)^2 \tag{2.14'}$$

where $b(T)$ is an arbitrary function of T. However, by comparing the above two equations (2.14) and (2.14′) we see that $b(T)$ must actually be a pure constant b, independent of T, and $f(x)$ is just $[(ax/c^2) + b]^2$. Our initial conditions then require $b = 1$, so that we have

$$\left(1 + \frac{aX}{c^2}\right)^2 = \left(1 + \frac{ax}{c^2}\right)^2 + \left(\frac{aT}{c}\right)^2 \tag{2.15}$$

From (2.7), (2.13) and (2.15) we obtain

$$\frac{(aT/c)}{1+(aX/c^2)} = v = -\frac{1}{\surd(g_{44})}\left(\frac{dx}{dt}\right)_X = \frac{aT/c}{\surd(g_{44})\,[1+(ax/c^2)]}\frac{dT}{dt}$$

From Equations (2.6), (2.7), (2.8) and the assumption that g_{44} does not depend on t, one can show that

$$g_{44} = \left(1+\frac{ax}{c^2}\right)^2 \tag{2.16}$$

The above equation in dT/dt can be integrated, and with the initial condition

$$t=0 \qquad \text{when } T=0$$

we obtain

$$\frac{2at}{c} = \ln\left(1+\frac{aX}{c^2}+\frac{aT}{c}\right) - \ln\left(1+\frac{aX}{c^2}-\frac{aT}{c}\right) \tag{2.17}$$

Using (2.15) in (2.17), we obtain

$$+\frac{1}{[1+(ax/c^2)]}\left(\frac{dx}{dt}\right)_X = -\frac{aT/c}{\surd\{[1+(ax/c^2)]^2+(aT/c)^2\}} = -\tanh\frac{at}{c} \tag{2.18}$$

and (2.7) becomes

$$\left(\frac{dX}{dT}\right)_x = \frac{aT/c}{1+(aX/c^2)} = \tanh\frac{at}{c} \tag{2.18a}$$

In (2.18a), since x is held fixed, t is also the proper time τ in S'.

Equations (2.15) and (2.17) now transform the metric

$$ds^2 = -dX^2 + dT^2 \tag{2.19}$$

into

$$ds^2 = -dx^2 + \left(1+\frac{ax}{c^2}\right)^2 dt^2 \tag{2.20}$$

In (2.20), for a given constant a, x is restricted to the region $x > -(c^2/a)$. It is seen from (2.13) and (2.15) that the limiting value $x = -(c^2/a)$ corresponds to $v = c$ beyond which v should not pass.

For convenience, we write three consequences of equations (2.15) and (2.17), namely,

$$\frac{aT}{c} = \left(1+\frac{aX}{c^2}\right)\tanh\left(\frac{at}{c}\right) \tag{2.21}$$

$$\frac{aT}{c} = \left(1+\frac{ax}{c^2}\right)\sinh\left(\frac{at}{c}\right) \tag{2.22}$$

$$1+\frac{aX}{c^2} = \left(1+\frac{ax}{c^2}\right)\cosh\frac{at}{c} \tag{2.23}$$

Equation (2.15) describes the motion of a fixed point p in S' from the standpoint of S. It is the so-called hyperbolic motion.

Equation (2.23) describes the motion of a fixed point p in S from the standpoint of S'.

If S' is accelerated along the $-X$ direction, we have only to replace a in all the equations (2.12)–(2.23) by $-a$, and obtain

$$\left(1-\frac{a(x-x_0)}{c^2}\right)^2=\left(1-\frac{a(X-X_0)}{c^2}\right)^2-\frac{a^2}{c^2}(T-T_0)^2 \tag{2.24}$$

$$\frac{a(T-T_0)}{c}=\left(1-\frac{a(X-X_0)}{c^2}\right)\tanh\frac{a(t-t_0)}{c} \tag{2.25}$$

$$\frac{a(T-T_0)}{c}=\left(1+\frac{a(X-X_0)}{c^2}\right)\sinh\frac{a(t-t_0)}{c} \tag{2.26}$$

$$1-\frac{a(X-X_0)}{c^2}=\left(1-\frac{a(x-x_0)}{c^2}\right)\cosh\frac{a(t-t_0)}{c} \tag{2.27}$$

where x_0, X_0, T_0, t_0 are constants to be determined by the appropriate initial conditions. In this case, equations (2.7) and (2.13) become

$$v=\left(\frac{dX}{dT}\right)_x=-\frac{a(T-T_0)}{1-(a/c^2)(X-X_0)} \tag{2.28}$$

$$v=-\frac{1}{1-[a(x-x_0)/c^2]}\left(\frac{dx}{dt}\right)_X=\left(\frac{dX}{dT}\right)_x=-\tanh a(t-t_0) \tag{2.28a}$$

At this point, it is of considerable interest to note that the transformation (2.15) and (2.17) are precisely that derived by Møller (1943) on completely different considerations. Møller starts from a static metric assumed to be

$$ds^2=-(dx^2+dy^2+dz^2)+g_{44}\,dt^2 \tag{2.8}$$

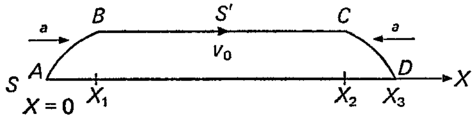
where $g_{44}=g_{44}(x)$. g_{44} is determined by the Einstein equations $R_{\mu\nu}=0$, which lead to $g_{44}=[1+(gx/c^2)]^2$, g being a constant. The curvature tensor $R^{\lambda}_{\mu\nu\sigma}$ for this metric vanishes, showing the space to be Euclidean. With the help of the equations of the geodesic, the transformation (2.15) and (2.17) is found. We have arrived at (2.15) and (2.17) on the basis of the local Lorentz transformation properties (2.3), (2.6), (2.7) with the assumption that in (2.7), g_{44} is a function of x but not of t.

3. *Resolution of the Clock Paradox*

We shall now study the clock problem as stated in (1.1) and (1.2) of Section 1, by treating the accelerated parts of the trip A–B, C–D–E, F–A, and A'–B', C'–D'–E', F'–A' by means of the accelerated motion described by equations (2.9)–(2.24) and (2.25)–(2.28) of the preceding section.

(A) *From the Standpoint of S*

Referring to the figure in (1.1), let the origins of the coordinate systems $S(X)$ and $S'(x)$ be coincident at $T = t = 0$.



(3.1)

The rocket $S'(x=0)$ moves according to (2.9), (2.7), (2.18)–(2.18a), (2.21)–(2.23). Part A–B: For the motion of the point $x = 0$ (fixed in S'), the t in (2.18a), (2.22), (2.23) becomes the proper time τ_0 in S' (i.e., the time registered by one and the same clock at $x = 0$ fixed in S'). When $x = 0$ reaches the velocity v_0 (relative to S frame) we have†

$$\tanh a\Delta\tau_1 = v_0 \tag{3.2}$$

$$aT_1 = \sinh(a\tau_{01}) = \frac{v_0}{\sqrt{(1-v_0^2)}} \tag{3.3}$$

$$1 + aX_1 = \cosh(a\tau_{01}) = \frac{1}{\sqrt{(1-v_0^2)}} \tag{3.4}$$

where T_1 is time, measured by the synchronised clocks attached to S, X_1 is the distance traversed by S' $(x=0)$ when it has reached the velocity v_0 (i.e., the part A–B). In the following, the subscript 1, 2, 3, ... refer to the parts A–B, B–C, C–D, etc. respectively of the trip. The same subscripts 1, 2, 3, ... are also used for the A'–B', B'–C', C'–D', etc. in the following section from the standpoint of S'.

To obtain the time interval ΔT_1 recorded by one clock fixed at $X = 0$ in S, let S' send light signals back to S. Let τ be the proper time in S'. Then‡

$$\Delta T_1 = \int_0^{\Delta\tau_1} \sqrt{\left(\frac{1+v}{1-v}\right)}\, d\tau \tag{3.5}$$

where by (3.2) $\Delta\tau_1 = \tanh^{-1} v_0$. Thus

$$\Delta T_1 = \frac{1}{a}\left[\frac{v_0}{\sqrt{(1-v_0^2)}} + \frac{1}{\sqrt{(1-v_0^2)}} - 1\right]$$

† In the following, we simplify writing by choosing the unit of time such that $c = 1$. All time, velocity, acceleration T, t, v, v_0, a are to be replaced by cT, ct, v/c, v_0/c, a/c^2, to convert to c.g.s. units.

‡ ΔT_1 only represents the time interval for the clock at $X = 0$ to intercept all the light signals sent to it by the clock attached to $x = 0$ within $\Delta\tau_1$. It does *not* really represent the time of travel of the rocket $(x = 0)$ from A to B as recorded by the clock at $X = 0$. The sum in (3.10) is, however, the total time interval for the whole trip of S', as recorded by one and the same clock at $X = 0$ in S.

From the symmetry of the situation, it is clear that for the part C–D,

$$\Delta T_3 = \Delta T_1 \tag{3.6}$$

and for the parts D–E, F–A,

$$\Delta T_4 = \int_0^{\Delta\tau_1} \sqrt{\left(\frac{1-v}{1+v}\right)}\, d\tau \tag{3.7}$$

$$= \frac{1}{a}\left[\frac{v_0}{\sqrt{(1-v_0^2)}} - \frac{1}{\sqrt{(1-v_0^2)}} + 1\right]$$

$$\Delta T_6 = \Delta T_4 \tag{3.8}$$

For the parts B–C, E–F, if $\Delta\tau_2 = \Delta\tau_5$ is the proper time intervals in S', the sum of the intervals for the arrivals of the signals sent back by S' during these intervals is

$$\Delta T_2 + \Delta T_5 = \Delta\tau_2 \sqrt{\left(\frac{1+v_0}{1-v_0}\right)} + \Delta\tau_5 \sqrt{\left(\frac{1-v_0}{1+v_0}\right)}$$

$$= \frac{2\Delta\tau_2}{\sqrt{(1-v_0^2)}} \tag{3.9}$$

Thus the total interval recorded by one single clock in S (at rest at $X=0$) from all signals sent back by S' during its round trip is

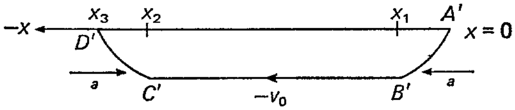
$$\sum_{j=1}^{6} \Delta T_j = \frac{4}{a}\frac{v_0}{\sqrt{(1-v_0^2)}} + \frac{2\Delta\tau_2}{\sqrt{(1-v_0^2)}} \tag{3.10}$$

The proper time (recorded by one clock) in S' for the whole round trip is

$$\Delta\tau_0 = 4\Delta\tau_{01} + 2\Delta\tau_2 = \frac{4}{a}\tanh^{-1} v_0 + 2\Delta\tau_2 \tag{3.11}$$

(B) *From the Standpoint of S'*

The fixed point $X=0$ in S moves in the negative x direction.



(3.12)

Part A'–B'. Let ΔT_1 be the proper time interval (registered by a clock at $X=0$ in S) for $X=0$ to reach the velocity $-v_0$ (relative to S'). From (2.18a),

$$a\Delta T_1 = \tanh\frac{a\tau_1}{c} = v_0 \tag{3.13}$$

and the distance traversed by $X=0$ during this interval is given by (2.23) (with $X=0$) and (2.18) with $\tanh at_1 = v_0$, i.e.,

$$1 + ax_1 = \frac{1}{\cosh at_1} = \sqrt{(1 - v_0^2)} \tag{3.14}$$

Parts B'–C' and E'–F'. Since the time interval for B'–C' or E'–F' in S' is $\Delta\tau_2$ in (3.8), S' would have deduced from the special theory of relativity that the combined intervals would have added to the proper time intervals of S the value $2\Delta T_2$ which can be calculated as follows. Let dT be an element of proper time in S. For the combined B'–C' and E'–F', light signals sent by S back to S' will reach S' in interval

$$\left[\Delta T_2 \sqrt{\left(\frac{1+v_0}{1-v_0}\right)} + \Delta T_5 \sqrt{\left(\frac{1-v_0}{1+v_0}\right)}\right] = \frac{2\Delta T_2}{\sqrt{(1-v_0^2)}}$$

This is now the time interval recorded by the clock in S', and is hence the proper time interval $2\Delta\tau_2$, i.e.,

$$2\Delta T_2 = \sqrt{(1 - v_0^2)}\, 2\Delta\tau_2 \tag{3.15}$$

Part C'–D'. During C'–D' and D'–E', S' would describe S as being acted on by a gravitational field a in the positive x-direction, the motion being described by equations (2.24)–(2.27). Equation (2.24) is

$$[1 - a(x - x_0)]^2 = [1 - a(X - X_0)]^2 - a^2(T - T_0)^2 \tag{2.24}$$

The constants x_0, X_0, T_0 are determined as follows. From the standpoint of S_1 [see (3.1), (3.2) and (3.8)], at C,

$$aT = \frac{v_0}{\sqrt{(1 - v_0^2)}} + \frac{a\Delta\tau_2}{\sqrt{(1 - v_0^2)}}, \qquad v = v_0 \tag{3.16}$$

and (2.28) leads to

$$-\frac{a\left[\dfrac{1}{\sqrt{(1-v_0^2)}}(v_0 + a\Delta\tau_2) - T_0\right]}{1 + aX_0} = v_0$$

At D,

$$T = \frac{2v_0}{\sqrt{(1 - v_0^2)}} + \frac{a\Delta\tau_2}{\sqrt{(1 - v_0^2)}},$$

$v = 0$ and

$$X = 2X_1 + \frac{v_0 \Delta\tau_2}{\sqrt{(1 - v_0^2)}}$$

[see (3.4)]. These lead to

$$X_0 = 2X_1 + \frac{v_0 \Delta\tau_2}{\sqrt{(1 - v_0^2)}}$$

$$aT_0 = \frac{2v_0}{\sqrt{(1 - v_0^2)}} + \frac{a\Delta\tau_2}{\sqrt{(1 - v_0^2)}} \tag{3.17}$$

$$x_0 = 0$$

From (2.28a),

$$\tanh a(t - t_0) = -v$$

the condition $v = v_0$ when $t = \Delta\tau_1 + \Delta\tau_2$ leads to

$$t_0 = 2\Delta\tau_1 + \Delta\tau_2 \tag{3.18}$$

Thus, equation (2.25) corresponding to the case (2.24) is now

$$a\left(T - \frac{2v_0}{\surd(1-v_0^2)} - \frac{\Delta\tau_2}{\surd(1-v_0^2)}\right) = \left[1 - a\left(X - 2X_1 - \frac{v_0\Delta\tau_2}{\surd(1-v_0^2)}\right)\right] \times \tanh(t - 2\Delta\tau_1 - \Delta\tau_2) \tag{3.19}$$

At D', $v = \tanh(t - 2\Delta\tau - \Delta\tau_2) = 0$ and the point $X = 0$ of S has the time $T = \Delta T_{A'-B'-C'-D'}$

$$\Delta T_{A'-B'-C'-D'} = \frac{2v_0}{\surd(1-v_0^2)} + \frac{\Delta\tau_2}{\surd(1-v_0^2)} \tag{3.20}$$

From the symmetry of the situation, it is clear that the time in S for the trip A'–B'–C'–D'–E'–F'–A' as seen (or, calculated) from the standpoint of S' is twice the $\Delta T_{A'-D'}$ in (3.20), i.e.,

Total time ΔT in $S =$

$$2\left(\frac{2v_0}{a\surd(1-v_0^2)} + \frac{\Delta\tau_2}{\surd(1-v_0^2)}\right) \tag{3.21}$$

which can be written

$$= 2(\Delta T_{A-B} + \Delta T_{B-C} + \Delta T_{C-D})$$

or,

$$a\Delta T = 2\left[v_0 + a\surd(1-v_0^2)\Delta\tau_2 + \frac{2v_0}{\surd(1-v_0^2)} + \frac{a\Delta\tau_2}{\surd(1-v_0^2)} - (v_0 + a\surd(1-v_0^2)\Delta\tau_2)\right] \tag{3.22}$$

the proper time in

$$S' = 4/a \tanh^{-1} v_0 + 2\Delta\tau_2 \tag{3.23}$$

Equation (3.21) shows complete agreement with the value given in (3.10) obtained from the standpoint of S. Equation (3.22) shows that the 'loss of time' during B'–C' and E'–F' by S in the view of S' [i.e., $2\surd(1-v_0^2)\Delta\tau_2$ in (3.15) compared with $2\Delta\tau_2/\surd(1-v_0^2)$ in (3.9)] is more than made up by the 'gain in time' by the S clock during C'–D', D'–E' when S is at a higher equivalent gravitational potential than S' [see (3.12)], and the clock of S is 'faster' on account of the factor $g_{44} = (1 - gx)^2$ in

$$ds^2 = dx^2 + (1 - gx)^2\, dt^2$$

(with $x = 0$ at A' and $ax = 2[\surd(1-v_0^2) - 1] - v_0\Delta\tau_2$ at D'). The smaller 'loss of time' of the clock of S during A'–B' (when S is at a lower gravitational potential, $ds^2 = -dx^2 + (1 + gx)^2 dt^2$) is also more than made up by the 'gain' during C'–D'. The total result is to bring the two reckonings of the proper time intervals, by S and S', of the round trip into *exact* agreement with each other.†

It is seen from the foregoing results that all the calculations are *exact*, and no approximations involving the assumption of making the accelerated parts A–B, C–D–E, F–A (or A'–B', C'–D'–E', F'–A') very short compared with the uniform relative motion part B–C, E–F (or B'–C', E'–F') have been made. In fact, as emphasised by Einstein as early as in the 1918 paper and brought out approximately by Tolman (1934) and exactly in (3.22) above that is precisely the accelerated parts that resolve the 'paradox'. Had one literally 'neglected' the accelerated parts, (3.10) and (3.22) would have become

$$\text{Total time in } S \text{ (as reckoned by } S) = \frac{2\Delta\tau_2}{\sqrt{(1-v_0^2)}}$$

$$\text{Total time in } S \text{ (as reckoned by } S') = 2\sqrt{(1-v_0^2)}\,\Delta\tau_2$$

On the other hand, had one done away entirely with the uniform relative motion (coasting of rocket) parts B–C, E–F (B'–C', E'–F'), the results (3.10) and (3.22) would have become:

	Standpoint of S	Standpoint of S'
Total proper time in S'	$\dfrac{4}{a}\tanh^{-1} v_0$	$\dfrac{4}{a}\tanh^{-1} v_0$
Total proper time in S	$\dfrac{4}{a}\dfrac{v_0}{\sqrt{(1-v_0^2)}}$	$\dfrac{2}{a}\left[v_0 + \dfrac{2v_0}{\sqrt{(1-v_0^2)}} - v_0\right]$

† The results (3.10), (3.11), (3.22), (3.23) above are a little more complete than those of Møller (1943) in that here S' starts out from rest and comes back at rest to S. There are differences in details between this and Møller's work. For example, we calculate the time intervals ΔT_1, ΔT_3, ΔT_4, ΔT_6 in (3.5)–(3.8) as recorded by one clock in S, and not Møller's times T', T'' which are not the proper times of one clock. Also, as remarked in Section 2 above, the starting points in the two works are different. In an application of Møller's work, Fock (1959) has obtained an erroneous conclusion.

Fock (1959) states that the time intervals recorded by the clocks A, B in S, S' are given by

$$\tau_A - \tau_B = \frac{v^2}{c^2}(\tfrac{1}{2}T - \tfrac{2}{3}t)$$

where $t = 2v/g$ is the time for the turning around part (C–D–E in (1.8) in the present article), and T = uniformly moving part (B–C) + (E–F) + t. Thus $\tau_A - \tau_B$ can be $\gtreqless 0$, in disagreement with the results of everyone else. This strange result arises from the error of the + sign in (62.09), which should have read $U = U_0 - g(x_1 - x)$. When this correction is made, one would have

$$\tau_A - \tau_B = \frac{v^2}{c^2}\frac{T}{4}$$

which is in agreement with the approximate result of (1.8) of Tolman and others.

Here is, of course, exact agreement between the reckonings of the total proper time intervals from the standpoints of both frames.

In the calculations above, we had employed the accelerated motion represented by equations (2.15) and (2.17) [or, (2.24) and (2.25)] which correspond to motion under a time-independent field. The result, however, is in fact quite general since from $S'(x,t)$, one can carry out any arbitrary coordinate transformation to a frame S'',

$$x'' = x''(x,t), \qquad t'' = t''(x,t)$$

which will lead in general to

$$ds^2 = \sum g_{ij}\, dx_i''\, dx_j''$$

where the g_{ij} are functions of x'' and t'' and hence no longer static. The space is, however, Euclidean. The motions of S'' relative to S can be quite arbitrary and very complicated, but the description can be reduced to that of S' by the transformation above so that in a sense the treatment of the clock problem by means of S' has covered a whole (infinite number) class of accelerated motions relative to S. This class of accelerated motions has not brought in any curved space properties in the sense of Einstein's general theory of relativity.

The present work has thus treated and resolved the clock problem without having really made recourse to Einstein's theory of gravitation involving curved space. This is worth noting in view of the usual statement in the literature that an exact treatment of the clock problem (i.e., to all orders of v_0/c) calls for the general theory of relativity.

4. *General Remarks on the 'Clock Paradox' Problem*

We are now in a position to summarise what we believe is relevant in the clock problem in the relativity theory.

(1) Invariance of proper time under coordinate transformations. In the theory of relativity (special and general),

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, d\chi_\mu\, d\chi_\nu \\ &= \text{invariant} \end{aligned} \tag{4.1}$$

in each group of coordinate transformations (the group in flat space-time which includes the Lorentz group, and the general group in curved space-time). Thus for a given world line C between two world points P_1 and P_2, the proper time interval

$$\Delta\tau \int_{P_1}^{P_2} ds \text{ along } C = \text{invariant} \tag{4.2}$$

i.e., has the same value in all frames satisfying (4.1)

(2) Proper time intervals between two world-points along different world lines.

Consider a given field $g_{\mu\nu} = g_{\mu\nu}(x_1, x_2, x_3, x_4)$. The motion of a particle from one world point P_1 to another P_2 is uniquely given by the geodesic C

$$\delta \int_1^2 ds = 0$$

Other paths C_1, C_2 joining P_1 and P_2 will not correspond to the 'free' motion in the field $g_{\mu\nu}$, but will correspond to motions under agencies other than the field representative by $g_{\mu\nu}$, and

$$\int_{1\,C_1}^{2} ds \neq \int_{1\,C}^{2} ds \tag{4.3}$$

which follows from the definition of the geodesic.

(3) For a given field $g_{\mu\nu}$, between two arbitrarily given points P_1 and P_2, there is one and only one geodesic.

From a point P_1, there are ∞^3 geodesics issuing in all directions. Of these, one, say C, goes through P_2. Suppose another, say C_1, makes an angle θ with C at P_1. Since a geodesic is a line generated by an infinitesimal vector in a continuous series of infinitesimal parallel displacements, and since the angle between two vectors is invariant under parallel displacements, it follows that in general two geodesics from a given point P_1 cannot intersect at another arbitrarily chosen point P_2. (In the case of a spherical surface, geodesics from a point P intersect at the antipode of P only.)

(4) The clock paradox. Let S and S' be two (material) frames whose coordinates transform according to (4.1). Let us follow Einstein's argument in Section 1, namely, from the standpoint of S' (the rocket), S undergoes a series of free falls in certain universal gravitation fields during $A'B'$, $C'D'$, $D'E'$, $F'A'$ and coasting $B'C'$, $E'E'$ in (1.2). Suppose these fields are represented by a field $g_{\mu\nu}$. From the standpoint S', the frame S passes from the initial point P_1 to P_2 (in 4-space) along the geodesic of $g_{\mu\nu}$, but S' itself passes from P_1 to P_2 along a pure-time trajectory since S' has been held fixed (at rest) by means of some *external* agency. Thus the world line of S' is not a geodesic of $g_{\mu\nu}$. In general, the proper time intervals along the two world lines between P_1 and P_2 are different, according to (4.3).

The above result is general, holding for curved space as well as for flat space. It is possible to make an explicit and exact calculation of the proper time intervals in the flat space case, using the spirit of the general theory of relativity (acceleration represented by a $g_{\mu\nu}$ field). This has been done by Møller (1943), and in the present work (Sections 2 and 3).

(5) Let $S(X, Y, Z, T)$ be a strictly inertial frame, i.e., frame in flat space-time, and $S'(x, y, z, t)$ be a frame in a curved space-time (i.e., in a gravitational field in Einstein's theory). Then there is no coordinate transformation which transforms $ds^2 = -(dx^2 + dY^2 + dZ^2) + dT^2$ into $ds^2 = \sum g_{\mu\nu} dx_\mu dx_\nu$ with the curvature tensor $R^\lambda_{\mu\nu\sigma} \neq 0$. In this case there is no invariant ds^2 and there is no exact (only approximate) connection between the space-time

description in S and that in S'. One can no longer compare $d\tau_0$ in S and the $d\tau_0$ in S', and the 'clock paradox' does not have any clear and exact meaning.

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